Noncollapsing solution below r_c for a randomly forced particle

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We show that a noncollapsing solution below r_c can be constructed for the dynamics of randomly forced particle interacting with a dissipating boundary. The scaling analysis predicts a divergent collision rate at the boundary for the noncollapsing solution. This prediction is tested numerically.

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Recently opposing viewpoints appeared in the literature [1–3] regarding the localization properties of a one-dimensional particle subject to an uncorrelated random force interacting with a dissipating boundary. The equation describing the particle dynamics is

$$\frac{d^2x}{dt^2}=\eta(t) \tag{1}$$

where $\eta(t)$ is white Gaussian noise, $\langle \eta(t)\eta(t')\rangle = 2\delta(t-t')$. The dissipating boundary condition is set such that the particle approaching with velocity $-u$, $(u>0)$, at the boundary $x=0$ is reflected with velocity ru, $r<1$. The problem was explored further with various techniques [4–6], which have obtained the same critical value for the dissipation parameter r_c and the persistence exponent. It is an unresolved problem, the absence of the collapsing behavior in numerical simulations [3].

In this paper, we propose a solution of this paradox. We show that for $r<r_c$ one can construct a constant mass (noncollapsing) solution starting from the collapsing one. In our approach, we use the Fokker-Planck equation (FPE) description of the process (1). The FPE associated with the Langevin equation (1) is

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial u^2}+u\frac{\partial}{\partial x}\right)P(x,u,t)=0 \tag{2}$$

with the dissipating boundary condition

$$P(0,-u)=r^2P(0,ru), \quad u>0, \tag{3}$$

and the initial condition $P(x,u,t=0)=\delta(x-x_0)\delta(u-u_0)$.

In Ref. [5], it was shown that the the general solution of this problem has the following integral form:

$$\begin{aligned}P(x,u;x_0,u_0;t)&=P_0(x,u;x_0,u_0;t)\\&\quad+\int_0^t dt_1\int_0^\infty du_1u_1P(x,u;0,ru_1;t-t_1)\\&\quad\times P_0(0,-u_1;x_0,u_0;t_1),\end{aligned} \tag{4}$$

where $P_0(x,u;x_0,u_0;t)$ is the solution of the FPE (2) with absorbing boundary at $x=0$ [7].

Burkhardt has shown in Ref. [4] that a collapsing solution with an algebraic temporal decay can be found for the Eq. (2). The surviving probability $Q(x_0,u_0,t)=\int dx dv P(x,u;x_0,u_0,t)$ behaves asymptotically as

$$Q(0,-u,t)\approx 2\sin\left[\frac{\pi}{6}(1-4\phi)\right]\left(\frac{u^2}{t}\right)^{\phi}, \quad u>0 \tag{5}$$

$$Q(0,u,t)\approx\left(\frac{u^2}{t}\right)^{\phi}, \quad u>0. \tag{6}$$

For the collapsing solution $(\phi>0)$ we have $Q(0,0,t)=0$, that is, the origin of the phase space $(x=0,\ u=0)$ is an absorbing point for the random particle. With this observation the collapsing behavior can also be obtained numerically. Using a discretization of Eq. (1) together with the absorbing prescription at the origin, the collapsing behavior is obtained with the persistence exponent in perfect agreement with theoretical result of Ref. [4], see Fig. 1. In numerical simulation the particle was absorbed if its velocity after the collision with the boundary was smaller than $\sqrt{\Delta t}$, where Δt is the time integration step. As Δt decreases the collapsing behavior is preserved signaling the existence of the collapsing behavior in the continuum limit.

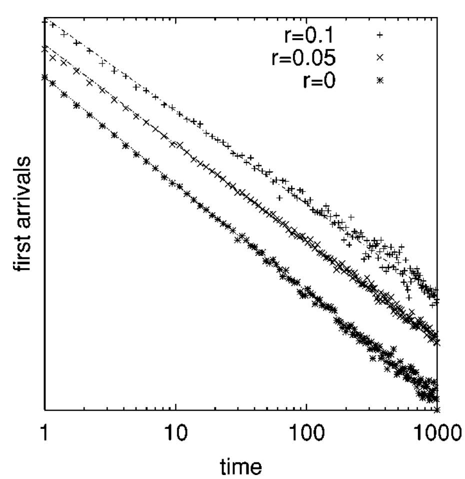

FIG. 1. First return distribution for $r=0.1,0.05,0$ ($r_c\approx 0.163$). The lines have the theoretical exponent found in Ref. [4].

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On the other hand, Eqs. (5), (6) accept the solution $\phi = 0$ at any value of the dissipation coefficient r—case in which the collapse does not occur since Q is constant. We make the second observation that Eq. (4) is not in contradiction with a solution that has constant mass, that is, $\int_0^\infty dx \int_{-\infty}^\infty du P(x,u;x_0,u_0;t) = 1$. Indeed if one integrates the over x and u the right hand side of Eq. (4) is the conservation law for the particle in case of the absorbing solution. The reason that allows the existence of more than one solution for this problem is that the dissipating condition $P(0,-u) = r^2 P(0,ru)$ does not specify uniquely the solution on the boundary $x=0$, but is just a condition that the solution must obey on the boundary $x=0$. The continuity condition for the solution asks that $P(0,u\to 0)\to 0$ or ∞.

We can construct a noncollapsing solution starting from the collapsing one. Let us start from the FPE with a source term

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial u^2} + u\frac{\partial}{\partial x}\right) P(x,u,t) = f(t)\,\delta(x-\epsilon)\,\delta(u), \qquad (7)$$

where $\epsilon > 0$ and $f(t)$ is at the moment an arbitrary function of time to be determined from the conserving mass condition. The solution of the above equation that satisfies the boundary condition (3) is

$$G_r(x,u;x_0,u_0;t) = P_r(x,u;x_0,u_0;t) + \int_0^t dt_1 P_r(x,u;\epsilon,0;t-t_1) f(t_1), \qquad (8)$$

where $P_r(x,u;x_0,u_0;t)$ is the collapsing solution of the problem at given $r<r_c$. After time Laplace transform, we have

$$G(x,u;x_0,u_0;s) = P_r(x,u;x_0,u_0;s) + P_r(x,u;\epsilon,0;s) f(s). \qquad (9)$$

We can choose $f(s)$ such that the mass of G is constant. We see that the choice

$$f(s) = \frac{\dfrac{1}{s} - \displaystyle\int_0^\infty dx_1 \int_0^\infty du_1 P_r(x_1,u_1;x_0,u_0;s)}{\displaystyle\int_0^\infty dx_1 \int_{-\infty}^\infty du_1 P_r(x_1,u_1;\epsilon,0;s)} \qquad (10)$$

gives us the needed solution. For any $\epsilon>0$ the solution Eq. (9) satisfies the boundary condition since we use P_r, and conserves the mass by construction. In the limit $\epsilon\to 0$ it satisfies also the initial condition $\delta(t)\delta(x-x_0)\delta(u-u_0)$. We make the observation that $f(s)=0$ for $r>r_c$ as $P_r(x,u;x_0,u_0;t)$ has a constant mass. It remains to be shown that the limit $\epsilon\to 0$ exists. For the case $r=0$ the asymptotic expressions for $P_r(x,u;x_0,u_0;s)$ and $\int_0^\infty \int_{-\infty}^\infty du P_r(x,u;x_0,u_0;s)$ were obtained in Ref. [7] and one can see explicitly that the above limit exists.

We can obtain the behavior of the collision rate at the origin for small ϵ using the scaling properties of the solution of the FPE. Equation (2) gives

$$P(x,u;x_0,u_0;s) = \lambda^2 P(\lambda^3 x, \lambda u; \lambda^3 x_0, \lambda u_0; \lambda^{-2} s). \qquad (11)$$

The collision rate R_{coll} is given by the following relations:

$$\begin{aligned} R_{\mathrm{coll}}(\epsilon;s) &= \int_{-\infty}^0 du\, u G(0,u;x_0,u_0;s) \\ &= \int_{-\infty}^0 du\, u P_r(0,u;x_0,u_0;s) \\ &\quad + \frac{\displaystyle\int_{-\infty}^0 du\, u P_r(0,u;\epsilon,0;s)}{\displaystyle\int_0^\infty dx \int_0^\infty du\, P_r(x,u;\epsilon,0;s)} \\ &\quad \times \left[\frac{1}{s} - \int_0^\infty dx \int_0^\infty du\, P_r(x,u;x_0,u_0;s)\right]. \end{aligned}$$

Using the scaling property, [Eq. (11)], with $\lambda = \epsilon^{-1/3}$ we have for the term depending on ϵ:

$$\frac{J(\epsilon;s)}{Q(\epsilon;s)} = \frac{\displaystyle\int_{-\infty}^0 du\, u P_r(0,u;\epsilon,0;s)}{\displaystyle\int_0^\infty dx \int_0^\infty du\, P_r(x,u;\epsilon,0;s)} = \frac{\tilde{J}(\epsilon^{2/3}s)}{\epsilon^{2/3}\tilde{Q}(\epsilon^{2/3}s)} \approx \epsilon^{-2\phi/3}, \quad \epsilon \ll 1, \qquad (12)$$

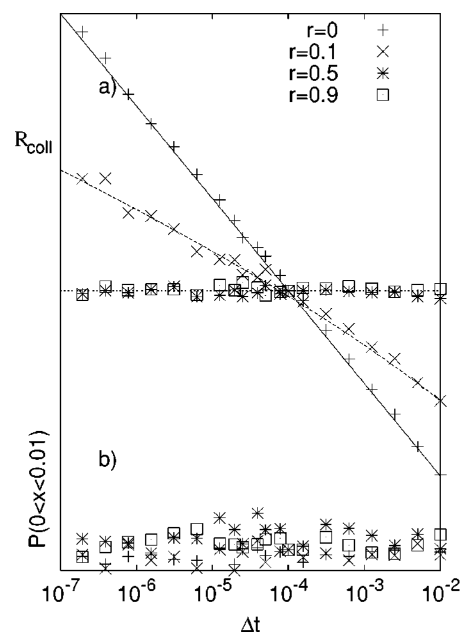

FIG. 2. (a) Collision rate at $x=0$ function of integration time step Δt at various values of the restitution coefficient r. We see that for $r>r_c\approx 0.163$ the collision rate is independent of Δt whereas diverges like $(\Delta t)^{-\phi}$ for $r<r_c$. The lines plot the theoretical prediction with $\phi=0.25$ $(r=0), \approx 0.087$ $(r=0.1), =0$ $(r\geqslant r_c)$. For $r=0.1$, we considered also the subleading correction. (b) Probability to find the particle in the interval (0,0.01) function of Δt. The quantity is constant for both $r<r_c$ and $r>r_c$. The graphs were displaced vertically for clarity.

where we have used that $\tilde{J}(s) \approx 1 - cs^{\phi}$ and $\tilde{Q}(s) \approx s^{-1+\phi}$ for $s \ll 1$.

Equation (12) shows that the conserving solution has divergent collision rate at the origin. This implies that the current density $uG(0,u;x_0,y_0;s)$ is nonintegrable, as was first noted in Ref. [8].

The prediction of a divergent collision rate can be checked numerically. If we integrate the Langevin equation with a finite time step Δt, then the particle is injected at $\epsilon \approx (\Delta t)^{3/2}$, where ϵ is its velocity that is very small. Consequently, the number of bounces at the origin must diverge as $(\Delta t)^{-\phi}$ as $\Delta t \to 0$. Indeed Fig. 2 shows a perfect validation of this prediction. In the same figure we have plotted the probability that the particle stays between $x=0$, $x=0.01$. We see that this this probability is independent of the integration step for each value of the restitution parameter r. This means that there is no collapsing behavior. The particle is attracted by the wall, performs an infinite number of collisions (in the limit $\Delta t \to 0$) and with probability 1 is injected back into the domain $x>0$.

Now we can see that the question ''what the particle does if we put it at $x=0$, $u=0$?'' is indeterminate for $r<r_c$. The general solution in this case is

$$P = qG_r(x,u,x_0,u_0,t) + (1-q)P_r(x,u,x_0,u_0,t), \quad 0 \leq q \leq 1, \tag{13}$$

and one has to specify q for the answer.

In conclusion, we have shown that the collapsing behavior can be found numerically if one notice that the collapsing solution has an absorbing point into the origin, hence it must be enforced in the simulation.

We have constructed a noncollapsing solution for the case $r<r_c$. This is possible since the absorbing boundary condition $P(0,-u) = r^2 P(0,ru)$ allows for two functions as boundary condition. One goes to zero as $u \to 0$ and the other one diverges to ∞ as $u \to 0$.

In terms of Brownian paths, we propose the following picture: for $r>r_c$ the probability to touch the origin of the phase space starting from any other point is zero, similar to simple diffusion in two or more dimensions, thus the collapsing solution is forbidden. When $r<r_c$ the diffusing particle touches the origin with probability 1 and if the path is set to terminate there the collapse occur. If the path is not set to terminate the particle is sent back into the domain $x>0$ after an infinite number of collision with the boundary. The weight of the paths leaving the origin without touching the boundary is zero in the continuum limit but they give a finite contribution because they are sampled an infinite number of times.

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